Mathematical Communications **13**(2008), 1-19

Category descriptions of the S_n- and S-equivalence

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Abstract. *By reducing the Mardešić S-equivalence to a finite case, i.e. to each $n \in \{0\} \cup \mathbb{N}$ separately, the authors recently derived the notion of S_n-equivalence of compacta. In this paper an additional notion of S_n^+-equivalence is introduced such that S_n^+ implies S_n and S_n implies S_{n-1}^+. The implications $S_1^+ \Rightarrow S_1 \Rightarrow S_0^+ \Rightarrow S_0$ as well as $Sh \Rightarrow S \Rightarrow S_1$ are strict. Further, for every $n \in \mathbb{N}$, a category $\underline{\mathcal{A}}_n$ and a homotopy relation on its morphism sets are constructed such that the mentioned equivalence relations admit appropriate descriptions in the given settings. There exist functors of $\underline{\mathcal{A}}_{n'}$ to $\underline{\mathcal{A}}_n$, $n \leq n'$, keeping the objects fixed and preserving the homotopy relation. Finally, the S-equivalence admits a category characterization in the corresponding sequential category $\underline{\mathcal{A}}$.*

Key words: *compactum, ANR, shape, S-equivalence, S_n-equivalence, category*

AMS subject classifications: Primary 55P55; Secondary 18A32

Received June 11, 2007 Accepted November 15, 2007

1. Introduction

A few decades ago S. Mardešić [4] introduced an equivalence relation between metric compacta, called the S-equivalence. The corresponding classification is strictly coarser than the shape type classification [2], [3], [7], [8]. Moreover, the S-equivalence on compact ANR's and compact polyhedra coincides with the homotopy type classification. However, the mentioned relation, being defined only on the class of objects, was not supported by an appropriate associated theory. In other words, it was not clear whether the S-equivalence admits a category characterization by means of its isomorphisms (or at least a category description by means of its morphisms). Furthermore, if such a (full) characterization would exists, there should exist a functor relating the shape category and the new category.

The reason why the S-equivalence was, for example, the problem of the shape types of fibres of a shape fibration. In 1977 D. Coram and P.F. Duvall [1] introduced

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and studied the approximate fibrations between compact ANR's. These are a shape analogue of the standard (Hurewicz) fibrations. In 1978 Mardešić and T.B. Rushing [5] generalized approximate fibrations to shape fibrations between metric compacta. The following important question was asking for the answer (analogously to the same homotopy type of the fibres of a fibration): Whether all the fibres of a shape fibration (over a continuum) have the same shape type? In 1979 J. Keesling and Mardešić [3] gave a negative answer. However, Mardešić [4] had proved before that all those fibres are mutually S-equivalent. He had also proved that some shape invariant classes of compacta (FANR's, movable compacta, compacta having shape dimension $\leq n, \dots$) are actually S-invariant.

By uniformization of the S-equivalence, Mardešić and Uglešić [7] obtained a finer equivalence relation, called the S^*-equivalence, that admits a full category characterization. A quite different characterization of the S^*-equivalence was given by the authors [8].

The S-equivalence is defined by means of a certain condition depending on every $n \in \mathbb{N}$. Mardešić and Uglešić had noticed in [7] that it makes sense to consider "the finite parts" of this condition. By following this idea, the authors [9] have recently reduced the mentioned condition to the finite cases, i.e. to every $n \in \{0\} \cup \mathbb{N}$ separately. In that way they derived the notions of S_n-equivalences of compacta They proved that the S_2-equivalence strictly implies S_1-equivalence and that S_1-equivalence strictly implies S_0-equivalence. Further, the S_1-equivalence restricted to compacta having the homotopy types of ANR's coincides with the homotopy type classification. Similarly, the S_1-equivalence restricted to the class of all FANR's (compacta having the shapes of ANR's) coincides with the shape type classification. Finally, the shape class, the S-equivalence class and the S_2-equivalence class of an FANR coincide.

In this paper we have provided a category description for each S_n-equivalence relation as well as for the S-equivalence. To do this, we have first introduced the additional equivalence relations, called the S_n^+-equivalences, $n \in \{0\} \cup \mathbb{N}$, such that

$$S_0 \Leftarrow S_0^+ \Leftarrow S_1 \Leftarrow \cdots \Leftarrow S_n \Leftarrow S_n^+ \Leftarrow S_{n+1} \Leftarrow \cdots .$$

The implications $S_0 \Leftarrow S_0^+ \Leftarrow S_1 \Leftarrow S_1^+$ as well as $S_1 \Leftarrow S \Leftarrow Sh$ are strict.

Further, for every $n \in \mathbb{N}$, a category $\underline{\mathcal{A}}_n$ and an equivalence (homotopy) relation on its morphisms sets are constructed such that the S_n (S_n^+)-equivalence admits an appropriate description by means of the corresponding morphisms of $\underline{\mathcal{A}}_{m(n)}$ ($\underline{\mathcal{A}}_{m(n)^+}$). More precisely, by following the basic idea of [8], the objects of $\underline{\mathcal{A}}_n$ are all inverse sequences $\boldsymbol{X}$ of compact ANR's (or compact polyhedra); a morphism set $\underline{\mathcal{A}}_n(\boldsymbol{X}, \boldsymbol{Y})$ consists of all so-called free n-hyperladders $F_n = (f_{\boldsymbol{j}^n}) : \boldsymbol{X} \to \boldsymbol{Y}$; there exists a homotopy relation $F_n \simeq F_n'$ on each morphism set. (However, this equivalence relation is not compatible with the composition, so there is no appropriate quotient category.) There exist functors of $\underline{\mathcal{A}}_{n'}$ to $\underline{\mathcal{A}}_n$, $n \leq n'$, which keep the objects fixed and preserve the homotopy relations (Theorem 1.). If there exist an $F_n \in \underline{\mathcal{A}}_n(\boldsymbol{X}, \boldsymbol{Y})$ and a $G_n \in \underline{\mathcal{A}}_n(\boldsymbol{Y}, \boldsymbol{X})$ such that $G_n F_n \simeq 1_{\boldsymbol{X}n}$ and $F_n G_n \simeq 1_{\boldsymbol{Y}n}$ hold, then we say that $\boldsymbol{X}$ is *n-alike* $\boldsymbol{Y}$, denoted by $\boldsymbol{X} \overset{n}{\leftrightarrow} \boldsymbol{Y}$. If X and Y are compacta, then $X \overset{n}{\leftrightarrow} Y$ is defined by means of $\boldsymbol{X} \overset{n}{\leftrightarrow} \boldsymbol{Y}$, where $\boldsymbol{X}, \boldsymbol{Y}$ are associated with X, Y, respectively ($X = \lim \boldsymbol{X}$, $Y = \lim \boldsymbol{Y}$). Among other facts, we have proved the following ones (Theorems 3-6):

$$(S_{3n+1}(X) = S_{3n+1}(Y)) \Rightarrow (X \overset{2n+1}{\leftrightarrow} Y), \quad n \geq 0;$$
$$(S^+_{3n+2}(X) = S^+_{3n+2}(Y)) \Rightarrow (X \overset{2n+2}{\leftrightarrow} Y), \quad n \geq 0;$$
$$(X \overset{2n+1}{\leftrightarrow} Y) \Rightarrow (S_n(X) = S_n(Y)), \quad n \geq 0;$$
$$(X \overset{2n}{\leftrightarrow} Y) \Rightarrow (S^+_{n-1}(X) = S^+_{n-1}(Y)), \quad n \geq 1.$$

As a consequence (Corollary 1.)

$$((X \overset{n}{\leftrightarrow} Y) \wedge (Y \overset{n}{\leftrightarrow} Z)) \Rightarrow (X \overset{[\frac{n}{3}]}{\leftrightarrow} Z), \quad n \geq 3.$$

Finally, $S(X) = S(Y)$ if and only if X and Y are *alike*, $X \leftrightarrow Y$ (Corollary 2.), where $X \leftrightarrow Y$ is defined by means of $X \overset{n}{\leftrightarrow} Y$ for all $n \in \mathbb{N}$.

2. Preliminaries

Let $c\mathcal{M}$ denote the class of all compact metrizable spaces (compacta), and let $c\underline{\mathcal{M}}$ denote the class of all inverse sequences over $c\mathcal{M}$. By [4], Definition 1., two inverse sequences $\boldsymbol{X}, \boldsymbol{Y} \in c\underline{\mathcal{M}}$ are said to be *S-equivalent*, denoted by $S(\boldsymbol{Y}) = S(\boldsymbol{X})$, provided, for every $n \in \mathbb{N}$, the following condition is fulfilled:

$$(\forall j_1)(\exists i_1)(\forall i'_1 \geq i_1)(\exists j'_1 \geq j_1)(\forall j_2 \geq j'_1)(\exists i_2 \geq i'_1)\cdots$$
$$\cdots(\forall i'_{n-1} \geq i_{n-1})(\exists j'_{n-1} \geq j_{n-1})(\forall j_n \geq j'_{n-1})(\exists i_n \geq i'_{n-1})$$

and there exist mappings

$$f_k \equiv f^n_{j_k} : X_{i_k} \to Y_{j_k}, k = 1, \ldots, n,$$

and

$$g_k \equiv g^n_{i'_k} : Y_{j'_k} \to X_{i'_k}, k = 1, \ldots, n-1,$$

making the following diagram

$$\begin{array}{ccccccccc} X_{i_1} & \leftarrow & X_{i'_1} & \leftarrow \cdots \leftarrow & X_{i'_{n-1}} & \leftarrow & X_{i_n} \\ \downarrow f_1 & & \uparrow g_1 & \cdots & \uparrow g_{n-1} & & \downarrow f_n \\ Y_{j_1} & \leftarrow & Y_{j'_1} & \leftarrow \cdots \leftarrow & Y_{j'_{n-1}} & \leftarrow & Y_{j_n} \end{array} \tag{D}$$

commutative up to homotopy. Two compacta X and Y are said to be *S-equivalent*, denoted by $S(Y) = S(X)$, provided there exists a pair (equivalently, for every pair) of limits $\boldsymbol{p} : X \to \boldsymbol{X}$ and $\boldsymbol{q} : Y \to \boldsymbol{Y}$ of inverse sequences consisting of compact ANR's such that $S(\boldsymbol{Y}) = S(\boldsymbol{X})$ (see [4], Remarks 1. and 2. and Definition 2.). If $\boldsymbol{p} : X \to \boldsymbol{X}$ is the limit, then we also say that $\boldsymbol{X}$ is *associated* with X.

If compacta X and Y have the same shape [6], $Sh(Y) = Sh(X)$, then $S(Y) = S(X)$. There exist compacta X and Y such that $S(Y) = S(X)$ and $Sh(Y) \neq Sh(X)$ (see [3], [2], [7]).

If the choice of indices i_k and j'_k does not depend on a given $n \in \mathbb{N}$ (while the mappings still depend on n, i.e. $f_k \equiv f^n_{j_k} : X_{i_k} \to Y_{j_k}$ and $g_k \equiv g^n_{i'_k} : Y_{j'_k} \to X_{i'_k}$), then the S-equivalence becomes the S^**-equivalence* (see [7], Definitions 6.-9., and [8], Lemmata 4. and 5.). There exists a pair X, Y of compacta such that $S^*(Y) = S^*(X)$ and $Sh(Y) \neq Sh(X)$ (see [7], [8]). However, we have no example yet which could show that the S^*-equivalence is indeed strictly finer than the S-equivalence.

According to [9], given an $n \in \mathbb{N}$, let us denote the above condition, relating $\boldsymbol{Y}$ to $\boldsymbol{X}$, by (D_{2n-1}). Further, let us denote by (D_{2n}) the following extension of (D_{2n-1}):

$$(\forall j_1)(\exists i_1)(\forall i'_1 \geq i_1)(\exists j'_1 \geq j_1) \cdots$$
$$\cdots (\forall j_n \geq j'_{n-1})(\exists i_n \geq i'_{n-1})(\forall i'_n \geq i_n)(\exists j'_n \geq j_n)$$

and there exist mappings

$$f_k \equiv f^n_{j_k} : X_{i_k} \to Y_{j_k},\ g_k \equiv g^n_{i'_k} : Y_{j'_k} \to X_{i'_k}, k = 1, \dots, n,$$

making diagram (D), extended by adding one rectangle, commutative up to homotopy.

It is obvious that (relating $\boldsymbol{Y}$ to $\boldsymbol{X}$), for each $m \in \mathbb{N}$,

$$(D_{m+1}) \Rightarrow (D_m).$$

Given any $\boldsymbol{X}, \boldsymbol{Y} \in c\underline{\mathcal{M}}$ and $n \in \{0\} \cup \mathbb{N}$, let $S_n(\boldsymbol{X}, \boldsymbol{Y})$ denote condition (D_{2n+1}) relating $\boldsymbol{Y}$ to $\boldsymbol{X}$. Further, let $S^+_n(\boldsymbol{X}, \boldsymbol{Y})$ denote condition (D_{2n+2}) relating $\boldsymbol{Y}$ to $\boldsymbol{X}$. It is clear that, for every $n \in \mathbb{N} \cup \{0\}$, the following assertions hold (see also Lemma 1. of [9]):

$$S_{n+1}(\boldsymbol{X}, \boldsymbol{Y}) \Rightarrow (S^+_n(\boldsymbol{X}, \boldsymbol{Y}) \wedge S^+_n(\boldsymbol{Y}, \boldsymbol{X}));$$
$$S^+_n(\boldsymbol{X}, \boldsymbol{Y}) \Rightarrow (S_n(\boldsymbol{X}, \boldsymbol{Y}) \wedge S_n(\boldsymbol{Y}, \boldsymbol{X}));$$
$$((\forall n \in \{0\} \cup \mathbb{N})\ S_n(\boldsymbol{X}, \boldsymbol{Y})) \Leftrightarrow ((\forall n \in \{0\} \cup \mathbb{N})\ S_n(\boldsymbol{Y}, \boldsymbol{X})) \Leftrightarrow$$
$$\Leftrightarrow (S(\boldsymbol{Y}) = S(\boldsymbol{X})).$$

We shall now slightly modify and extend Definition 1. of [9] such the notion of the S_n-equivalence remains unchanged, while the S_{n+1}-domination becomes the S^+_n-domination. We want to introduce the (new) notions of S_n-domination and S^+_n-equivalence.

Definition 1. *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be inverse sequences of compacta and let $n \in \{0\} \cup \mathbb{N}$. Then $\boldsymbol{Y}$ is said to be S_n**-dominated by $\boldsymbol{X}$**, denoted by $S_n(\boldsymbol{Y}) \leq S_n(\boldsymbol{X})$, provided condition $S_n(\boldsymbol{Y}, \boldsymbol{X})$ holds; $\boldsymbol{Y}$ is said to be S_n**-equivalent to $\boldsymbol{X}$**, denoted by $S_n(\boldsymbol{Y}) = S_n(\boldsymbol{X})$, provided the both conditions $S_n(\boldsymbol{Y}, \boldsymbol{X})$ and $S_n(\boldsymbol{X}, \boldsymbol{Y})$ are fulfilled. Similarly and dually, $\boldsymbol{Y}$ is said to be S^+_n**-dominated** by $\boldsymbol{X}$, denoted by $S^+_n(\boldsymbol{Y}) \leq S^+_n(\boldsymbol{X})$, provided condition $S^+_n(\boldsymbol{X}, \boldsymbol{Y})$ holds; $\boldsymbol{Y}$ is said to be S^+_n**-equivalent to $\boldsymbol{X}$**, denoted by $S^+_n(\boldsymbol{Y}) = S^+_n(\boldsymbol{X})$, provided the both conditions $S^+_n(\boldsymbol{X}, \boldsymbol{Y})$ and $S^+_n(\boldsymbol{Y}, \boldsymbol{X})$ are fulfilled.*

If X and Y are compacta, then we define $S_n(Y) \leq S_n(X)$ and $S_n(Y) = S_n(X)$ ($S^+_n(Y) \leq S^+_n(X)$ and $S^+_n(Y) = S^+_n(X)$) provided $S_n(\boldsymbol{Y}) \leq S_n(\boldsymbol{X})$ and $S_n(\boldsymbol{Y}) =$

$S_n(\boldsymbol{X})$ ($S_n^+(\boldsymbol{Y}) \leq S_n^+(\boldsymbol{X})$ and $S_n^+(\boldsymbol{Y}) = S_n^+(\boldsymbol{X})$) respectively, for some, equivalently: any, compact ANR inverse sequences $\boldsymbol{X}$, $\boldsymbol{Y}$ associated with X, Y respectively.

Obviously, by definition,

$$(S_n(\boldsymbol{Y}) = S_n(\boldsymbol{X})) \Leftrightarrow (S_n(\boldsymbol{Y}) \leq S_n(\boldsymbol{X}) \wedge S_n(\boldsymbol{X}) \leq S_n(\boldsymbol{Y})),$$
$$(S_n^+(\boldsymbol{Y}) = S_n^+(\boldsymbol{X})) \Leftrightarrow (S_n^+(\boldsymbol{Y}) \leq S_n^+(\boldsymbol{X}) \wedge S_n^+(\boldsymbol{X}) \leq S_n^+(\boldsymbol{Y})).$$

Further, according to Lemma 2. of [9], the following implications hold:

$$(S_{n+1}(\boldsymbol{Y}) \leq S_{n+1}(\boldsymbol{X})) \Rightarrow (S_n^+(\boldsymbol{Y}) = S_n^+(\boldsymbol{X})),$$
$$(S_n^+(\boldsymbol{Y}) \leq S_n^+(\boldsymbol{X})) \Rightarrow (S_n(\boldsymbol{Y}) = S_n(\boldsymbol{X})).$$

Furthermore, $S(\boldsymbol{Y}) = S(\boldsymbol{X})$ if and only if, for every $n \in \{0\} \cup \mathbb{N}$, $S_n(\boldsymbol{Y}) = S_n(\boldsymbol{X})$ (or, equivalently, $S_n^+(\boldsymbol{Y}) = S_n^+(\boldsymbol{X})$). Analogous statements hold for compacta as well. Consequently, the following sequence of implications (of equivalences of compacta strictly coarser than the shape type classification - Sh) is established:

$$S_0 \Leftarrow S_0^+ \Leftarrow S_1 \Leftarrow \cdots \Leftarrow S_n \Leftarrow S_n^+ \Leftarrow S_{n+1} \Leftarrow \cdots \Leftarrow S \Leftarrow S^* \Leftarrow Sh.$$

Observe that the S_0-domination is a trivial relation, i.e. for every pair X ($\neq \varnothing$), Y of compacta, $S_0(Y) \leq S_0(X)$ holds. Also, $S_0(\varnothing) \leq S_0(\varnothing)$. The S_0-equivalence is the trivial equivalence relation, i.e. for every pair X, Y of nonempty compacta, $S_0(Y) = S_0(X)$ holds. Also, $S_0(\varnothing) = \{\varnothing\}$. The S_1-equivalence strictly implies S_0-equivalence (Theorem 1. and Example 1. of [9]), and the S_2-equivalence strictly implies S_1-equivalence (Theorem 2. and Example 2. of [9]). The S_0^+-equivalence is not trivial because $S_0^+(\{*\}) \leq S_0^+(\{*\} \sqcup \{*\})$ and $S_0^+(\{*\} \sqcup \{*\}) \nleq S_0^+(\{*\})$. Further, the S_1-equivalence strictly implies S_0^+-equivalence. Namely, by Theorem 6. and Example 3. of [9], $S_0^+(C) = S_0^+(L)$ and $S_1(C) \neq S_1(L)$, where C is the Cantor set and

$$L = \{\tfrac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\} \subseteq \mathbb{R}.$$

Moreover, by Theorems 1. and 5. of [9], every two compacta, such that each of them is shape dominated by the other, are S_0^+-equivalent. Example 1. of [9] shows that there exists a pair of mutually homotopy dominated compacta which are not S_1-equivalent. By Theorem 2. and Example 2. of [9], the S_1^+-equivalence strictly implies S_1-equivalence. Indeed, $S_1^+(L) \leq S_1^+(L \sqcup L)$, while $S_1^+(L \sqcup L) \nleq S_1^+(L)$. Finally, as we mentioned before, Sh strictly implies S^* (see [7] and [8]). Hence, the next implications are strict:

$$S_0 \Leftarrow S_0^+ \Leftarrow S_1 \Leftarrow S_1^+,\ S_1 \Leftarrow S \text{ and } S^* \Leftarrow Sh.$$

3. Construction of the categories and functors

We are following the basic idea for a "subshape" category construction described in Section 2. of [8]. However, in this setting we have to abandon the "uniformity" conditions for the morphisms and their homotopy relation (see Definitions 2.4. and 2.7. of [8]). We only need a slight control over the index functions. First of all, recall the notion of an n-ladder ([8], Definition 2.1.). For a given $n \in \mathbb{N}$ and any $j_1, \ldots, j_{n+1} \in \mathbb{N}$ such that $j_1 < \cdots < j_{n+1}$, the corresponding ordered $(n+1)$-tuple $(j_1, \ldots, j_{n+1})$ is denoted by $\boldsymbol{j}^n$. The set of all such $(n+1)$-tuples $\boldsymbol{j}^n$ is denoted by $\boldsymbol{J}(n)$. Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be inverse sequences of compact metric spaces, let $n \in \mathbb{N}$ and let $\boldsymbol{j}^n \in \boldsymbol{J}(n)$. An ordered pair (f, f_j) consisting of an increasing (index) function

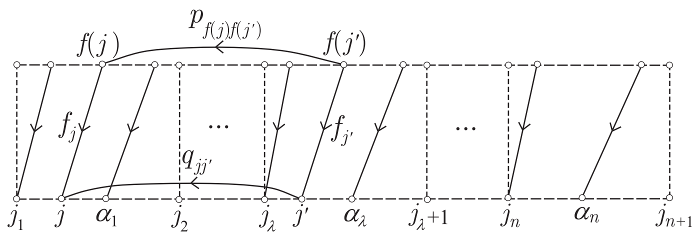
$$f : \bigcup_{\lambda=1}^{n} [j_\lambda, \alpha_\lambda]_{\mathbb{N}} \to [j_1, j_{n+1} - 1]_{\mathbb{N}},\ j_\lambda \leq \alpha_\lambda < j_{\lambda+1},$$

and of a set of mappings

$$f_j : X_{f(j)} \to Y_j,\ j \in \bigcup_{\lambda=1}^{n} [j_\lambda, \alpha_\lambda]_{\mathbb{N}},$$

is said to be an n**-ladder of** $\boldsymbol{X}$ **to** $\boldsymbol{Y}$ **over** $\boldsymbol{j}^n$, denoted by $f_{\boldsymbol{j}^n} : \boldsymbol{X} \to \boldsymbol{Y}$, provided the two following conditions are satisfied:

$$(\forall \lambda \in [1, n]_{\mathbb{N}})\ f(j_\lambda) \geq j_\lambda \wedge f(\alpha_\lambda) < j_{\lambda+1};$$
$$(\forall j, j' \in \bigcup_{\lambda=1}^{n} [j_\lambda, \alpha_\lambda]_{\mathbb{N}})\ j \leq j' \Rightarrow f_j p_{f(j)f(j')} \simeq q_{jj'} f_{j'}.$$



An n-ladder $f_{\boldsymbol{j}^n}$ having an *empty* λ*-block*, i.e. with no mapping for any $j \in [j_\lambda, j_{\lambda+1} - 1]_{\mathbb{N}}$, is allowed. We also admit the *empty* n*-ladder of* $\boldsymbol{X}$ *to* $\boldsymbol{Y}$ *over* $\boldsymbol{j}^n$, i.e. the empty set of mappings for a given $\boldsymbol{j}^n$.
If $f_{\boldsymbol{j}^n} : \boldsymbol{X} \to \boldsymbol{Y}$ and $g_{\boldsymbol{k}^n} = (g, g_k) : \boldsymbol{Y} \to \boldsymbol{Z}$ are n-ladders, then we **compose** them only in the case $\boldsymbol{j}^n = \boldsymbol{k}^n$ by using the ordinary rule, i.e.

$$g_{\boldsymbol{k}^n} f_{\boldsymbol{k}^n} \equiv u_{\boldsymbol{k}^n} = (u, u_k),$$

such that $u = fg$ (wherever it is defined) and $u_k = g_k f_{g(k)},\ k \in \bigcup_{\lambda=1}^{n} [k_\lambda, \gamma_\lambda]_{\mathbb{N}}$, $\gamma_\lambda \leq \beta_\lambda$. Clearly, $g_{\boldsymbol{k}^n} f_{\boldsymbol{k}^n} : \boldsymbol{X} \to \boldsymbol{Z}$ is an n-ladder of $\boldsymbol{X}$ to $\boldsymbol{Z}$ over $\boldsymbol{k}^n$. Notice that its λ-block is empty whenever the corresponding block of $f_{\boldsymbol{k}^n}$ or $g_{\boldsymbol{k}^n}$ is empty, or $g(k_\lambda) > \alpha_\lambda$. It is obvious that the composition of n-ladders is associative. Let $1_{\boldsymbol{X}\boldsymbol{i}^n}$ be the restriction of the identity morphism (of the inv-category $(c\mathcal{M})^{\mathbb{N}}$) $1_{\boldsymbol{X}} = (1_{\mathbb{N}}, 1_{X_i}) : \boldsymbol{X} \to \boldsymbol{X}$ to $\boldsymbol{i}^n \in \boldsymbol{J}(n)$. Clearly, $1_{\boldsymbol{X}\boldsymbol{i}^n}$ is an n-ladder of $\boldsymbol{X}$ to $\boldsymbol{X}$ over $\boldsymbol{i}^n$. Notice that $f_{\boldsymbol{j}^n} 1_{\boldsymbol{X}\boldsymbol{j}^n} = f_{\boldsymbol{j}^n}$ and $1_{\boldsymbol{X}\boldsymbol{i}^n} g_{\boldsymbol{i}^n} = g_{\boldsymbol{i}^n}$ hold for all n-ladders $f_{\boldsymbol{j}^n} : \boldsymbol{X} \to \boldsymbol{Y}$ and $g_{\boldsymbol{i}^n} : \boldsymbol{Z} \to \boldsymbol{X}$.

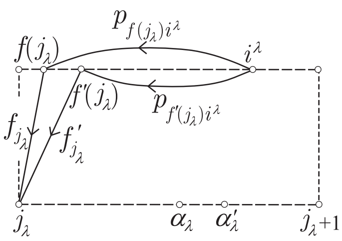
Let $n = 1$ and $\boldsymbol{j}^1 \in \boldsymbol{J}(1)$, and let $f_{\boldsymbol{j}^1}, f'_{\boldsymbol{j}^1} = (f', f'_j) : \boldsymbol{X} \to \boldsymbol{Y}$ be 1-ladders over the same $\boldsymbol{j}^1$. Then $f_{\boldsymbol{j}^1}$ is said to be **homotopic to** $f'_{\boldsymbol{j}^1}$ (compare Definition 2.3. of

[8]) provided they both are empty or there exists an $i^1 \in [j_1, j_2 - 1]_{\mathbb{N}}$ such that

$$f_{j_1} p_{f(j_1)i^1} \simeq f'_{j_1} p_{f'(j_1)i^1}.$$

In the general case of a pair of n-ladders the definition of being m-homotopic, $m \leq n$, is as follows: Let $n, m \in \mathbb{N} \cup \{\omega\}$, $m \leq n$, and let $f_{\boldsymbol{j}^n}, f'_{\boldsymbol{j}^n} : \boldsymbol{X} \to \boldsymbol{Y}$ be n-ladders over the same $\boldsymbol{j}^n$. Then $f_{\boldsymbol{j}^n}$ is said to be **m-homotopic to $f'_{\boldsymbol{j}^n}$**, denoted by $f_{\boldsymbol{j}^n} \simeq_m f'_{\boldsymbol{j}^n}$, provided, for every $\lambda \in [1, m]_{\mathbb{N}}$, the both $f_{\boldsymbol{j}^n}$ and $f'_{\boldsymbol{j}^n}$ have the λ-block empty or there exists an $i^\lambda \in [j_\lambda, j_{\lambda+1} - 1]_{\mathbb{N}}$ such that

$$f_{j_\lambda} p_{f(j_\lambda)i^\lambda} \simeq f'_{j_\lambda} p_{f'(j_\lambda)i^\lambda}.$$



Notice that $f_{\boldsymbol{j}^n} \simeq_{m'} f'_{\boldsymbol{j}^n}$ implies $f_{\boldsymbol{j}^n} \simeq_m f'_{\boldsymbol{j}^n}$ whenever $m \leq m'$. Clearly, the m-homotopy relation of n-ladders is an equivalence relation on the corresponding set. In the case of $m = n$, we simply write $f_{\boldsymbol{j}^n} \simeq f'_{\boldsymbol{j}^n}$.

Definition 2. *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be inverse sequences of compact metric spaces and let $n \in \mathbb{N}$. A **free n-hyperladder of $\boldsymbol{X}$ to $\boldsymbol{Y}$**, denoted by $F_n : \boldsymbol{X} \to \boldsymbol{Y}$, is any family $F_n = (f_{\boldsymbol{j}^n})$ of n-ladders $f_{\boldsymbol{j}^n} : \boldsymbol{X} \to \boldsymbol{Y}$ indexed by all $\boldsymbol{j}^n \in \boldsymbol{J}(n)$. The set of all free n-hyperladders F_n of $\boldsymbol{X}$ to $\boldsymbol{Y}$ is denoted by $\boldsymbol{L}_n(\boldsymbol{X}, \boldsymbol{Y})$.*

If $F_n = (f_{\boldsymbol{j}^n}) : \boldsymbol{X} \to \boldsymbol{Y}$ and $G_n = (g_{\boldsymbol{k}^n}) : \boldsymbol{Y} \to \boldsymbol{Z}$, $\boldsymbol{k}^n \in \boldsymbol{J}(n)$, are free n-hyperladders, then we **compose** them by composing the appropriate n-ladders $f_{\boldsymbol{j}^n}$ and $g_{\boldsymbol{k}^n}$ such that $\boldsymbol{j}^n = \boldsymbol{k}^n$. Hence,

$$G_n F_n \equiv U_n = (u_{\boldsymbol{k}^n}) : \boldsymbol{X} \to \boldsymbol{Z}$$

is a free n-hyperladder, where $u_{\boldsymbol{k}^n} \equiv g_{\boldsymbol{k}^n} f_{\boldsymbol{k}^n}$, $\boldsymbol{k}^n \in \boldsymbol{J}(n)$. Since the composition of n-ladders is associative, the composition of free n-hyperladders is associative. Notice that $1_{\boldsymbol{X}n} = (1_{\boldsymbol{X}\boldsymbol{i}^n})$, $\boldsymbol{i}^n \in \boldsymbol{J}(n)$, is the *identity n-hyperladder* on $\boldsymbol{X}$. Indeed,

$$F_n 1_{\boldsymbol{X}n} = (f_{\boldsymbol{j}^n})(1_{\boldsymbol{X}\boldsymbol{i}^n}) = (f_{\boldsymbol{j}^n} 1_{\boldsymbol{X}\boldsymbol{j}^n}) = (f_{\boldsymbol{j}^n}) = F_n,$$
$$1_{\boldsymbol{X}n} G_n = (1_{\boldsymbol{X}\boldsymbol{i}^n})(g_{\boldsymbol{i}^n}) = (1_{\boldsymbol{X}\boldsymbol{i}^n} g_{\boldsymbol{i}^n}) = (g_{\boldsymbol{i}^n}) = G_n$$

hold for all n-hyperladders $F_n : \boldsymbol{X} \to \boldsymbol{Y}$ and $G_n : \boldsymbol{Z} \to \boldsymbol{X}$. Thus, for every $n \in \mathbb{N}$, there exists a category $\underline{\mathcal{M}}_n$ having the object class $\mathrm{Ob}\,\underline{\mathcal{M}}_n$, which consists of all inverse sequences in $c\mathcal{M}$, and the morphism class $\mathrm{Mor}\,\underline{\mathcal{M}}_n$, which consists of all the morphism sets $\underline{\boldsymbol{L}}_n(\boldsymbol{X}, \boldsymbol{Y})$. Further, there exists the corresponding sequential category $\underline{\mathcal{M}} =$ “$(\underline{\mathcal{M}}_n)$” (see Section 5. below).

For every $n \in \mathbb{N}$, let $\underline{\mathcal{A}}_n$ denote the full subcategory of $\underline{\mathcal{M}}_n$ whose objects are all inverse sequences $\boldsymbol{X}$ in $cANR$ (compact ANR's) or $cPol$ (compact polyhedra), and let $\underline{\mathcal{A}} \subseteq \underline{\mathcal{M}}$ be the corresponding sequential (sub)category.

In order to define a certain equivalence (homotopy) relation on each set $\boldsymbol{L}_n(\boldsymbol{X},\boldsymbol{Y})$, let us first consider the simplest case $n=1$. Recall that $f_{\boldsymbol{j}^1} \simeq f'_{\boldsymbol{j}^1} : \boldsymbol{X} \to \boldsymbol{Y}$ means

$$(\exists i^1 \in [j_1, j_2 - 1]_{\mathbb{N}})\ f_{j_1} p_{f(j_1)i^1} \simeq f'_{j_1} p_{f'(j_1)i^1}.$$

Let $F_1 = (f_{\boldsymbol{j}^1}), F'_1 = (f'_{\boldsymbol{j}^1}) : \boldsymbol{X} \to \boldsymbol{Y}$ be a pair of free 1-hyperladders. Then F_1 is said to be **homotopic to** F'_1, provided every $j_1 \in \mathbb{N}$ admits an $i^1 \in \mathbb{N}$, $i^1 \geq j_1$, such that, for every $j_2 > i^1$, the corresponding 1-ladders $f_{\boldsymbol{j}^1} \in F_1$ and $f'_{\boldsymbol{j}^1} \in F'_1$ (assigned to the pair $\boldsymbol{j}^1 = (j_1, j_2) \in \boldsymbol{J}(1)$) are homotopic, $f_{\boldsymbol{j}^1} \simeq f'_{\boldsymbol{j}^1}$ with respect to the chosen index i^1. Briefly, $F_1 \simeq F'_1$ provided

$$(\forall j_1 \in \mathbb{N})(\exists i^1 \geq j_1)(\forall j_2 > i^1)\ \boldsymbol{f}_{\boldsymbol{j}^1} \simeq \boldsymbol{f}'_{\boldsymbol{j}^1}\ (\text{“up to } i^1\text{”}).$$

In the general case, the definition is as follows (compare Definition 2.7. of [8]):

Definition 3. *Let $n \in \mathbb{N}$ and let $F_n = (f_{\boldsymbol{j}^n}), F'_n = (f'_{\boldsymbol{j}^n}) : \boldsymbol{X} \to \boldsymbol{Y}$ be a pair of free n-hyperladders. Then F_n is said to be* ***homotopic to*** *F'_n, denoted by $F_n \simeq F'_n$, provided $F_n = F'_n$, or $F_n \neq F'_n$ and*

$$\begin{gathered}(\forall m \leq n)\\ (\forall j_1 \in \mathbb{N})(\exists i^1 \geq j_1)(\forall j_2 > i^1)\cdots(\forall j_m > i^{m-1})(\exists i^m \geq j_m)(\forall j_{m+1} > i^m)\\ (\forall j_{m+2} > j_{m+1})\ldots(\forall j_{n+1} > j_n)\end{gathered}$$

the corresponding n-ladders $f_{\boldsymbol{j}^n} \in F_n$ and $f'_{\boldsymbol{j}^n} \in F'_n$ are m-homotopic, $f_{\boldsymbol{j}^n} \simeq_m f_{\boldsymbol{j}^n}$ (“up to $i^1,\ldots,i^m$”).

It is readily seen that this homotopy relation is an equivalence relation on each set $\boldsymbol{L}_n(\boldsymbol{X},\boldsymbol{Y})$. The homotopy class $[F_n]$ of an $F_n \in \boldsymbol{L}_n(\boldsymbol{X},\boldsymbol{Y})$ is denoted by $\boldsymbol{F}_n : \boldsymbol{X} \to \boldsymbol{Y}$.

Remark 1. *The homotopy relation of free n-hyperladders is* not *compatible with the composition in the category $\underline{\mathcal{M}}_n$. Thus, although the quotient sets $\underline{\mathcal{M}}_n(\boldsymbol{X},\boldsymbol{Y})/\simeq$ exist, there is no appropriate quotient category.*

Theorem 1. *For every pair $n, n' \in \mathbb{N}$, $n \leq n'$, there exists a* restriction functor *$\underline{R}_{nn'} : \underline{\mathcal{M}}_{n'} \to \underline{\mathcal{M}}_n$ (which is* not *unique) keeping the objects fixed and preserving the homotopy relation. $\underline{R}_{nn}$ is the identity functor. Furthermore, for all $n \leq n' \leq n''$, there exist $\underline{R}_{nn'}$, $\underline{R}_{n'n''}$ and $\underline{R}_{nn''}$ such that $\underline{R}_{nn'}\underline{R}_{n'n''} = \underline{R}_{nn''}$. The same holds for the subcategories $\underline{\mathcal{A}}_n \subseteq \underline{\mathcal{M}}_n$, $n \in \mathbb{N}$.*

Proof. Let $n \leq n'$ and let $F_{n'} = (f_{\boldsymbol{j}^{n'}}) \in \underline{\boldsymbol{L}}_{n'}(\boldsymbol{X},\boldsymbol{Y})$. For every $\boldsymbol{j}^{n'} \in \boldsymbol{J}(n')$, let $f^{\boldsymbol{j}^{n'}}_{\boldsymbol{j}^n}$ denote the restriction of $f_{\boldsymbol{j}^{n'}} \in F_{n'}$ to a $\boldsymbol{j}^n \in \boldsymbol{J}(n)$. Let

$$\psi : \boldsymbol{J}(n) \to \boldsymbol{J}(n'),\ \psi(\boldsymbol{j}^n) = \boldsymbol{j}^{n'},$$

be an injective function such that

$$(\forall \lambda \in [1, n+1]_{\mathbb{N}})\ j'_\lambda = j_\lambda.$$

Notice that ψ induces a certain function $\Psi = \Psi_{\psi,\boldsymbol{X},\boldsymbol{Y}}$ of $\underline{\boldsymbol{L}}_{n'}(\boldsymbol{X},\boldsymbol{Y})$ to $\underline{\boldsymbol{L}}_n(\boldsymbol{X},\boldsymbol{Y})$ given by

$$F_{n'} = (f_{\boldsymbol{j}^{n'}}) \mapsto (f_{\boldsymbol{j}^n}) = F_n = \Psi(F_{n'}),$$

where $f_{\boldsymbol{j}^n} = f_{\boldsymbol{j}^n}^{\psi(\boldsymbol{j}^n)}$ for every $\boldsymbol{j}^n \in \boldsymbol{J}(n)$.
Observe that $\Psi(1_{\boldsymbol{X}^{n'}}) = 1_{\boldsymbol{X}^n}$. Moreover, since the composition of free hyperladders is defined coordinatewise (by indices), the following fact is obvious:

If $G_{n'}F_{n'} = U_{n'} \mapsto U_n$, $G_{n'} \mapsto G_n$ and $F_{n'} \mapsto F_n$, then $U_n = G_nF_n$.

It implies that $\Psi(G_{n'}F_{n'}) = \Psi(G_{n'})\Psi(F_{n'})$. Therefore, the injection ψ induces a functor $\underline{R}^{\psi}_{nn'} : \underline{\mathcal{M}}_{n'} \to \underline{\mathcal{M}}_n$, keeping the objects fixed, determined by $\underline{R}^{\psi}_{nn'}(F_{n'}) = \Psi(F_{n'})$. Let $F'_{n'} = (f'_{\boldsymbol{j}^{n'}}) \in \underline{\boldsymbol{L}}_{n'}(\boldsymbol{X},\boldsymbol{Y})$ be another free n'-hyperladder such that $F'_{n'} \simeq F_{n'}$, and let $\underline{R}^{\psi}_{nn'}(F'_{n'}) = F'_n = (f'_{\boldsymbol{j}^n}) \in \underline{\boldsymbol{L}}_n(\boldsymbol{X},\boldsymbol{Y})$. Then one readily sees that, for every $m \leq n$ $(\leq n')$, the corresponding homotopy condition of $F_{n'} \simeq F'_{n'}$ implies the analogous condition for $\underline{R}^{\psi}_{nn'}(F_{n'}) \simeq \underline{R}^{\psi}_{nn'}(F'_{n'})$. Thus, the functor $\underline{R}^{\psi}_{nn'} : \underline{\mathcal{M}}_{n'} \to \underline{\mathcal{M}}_n$ preserves the homotopy equivalence relation of free hyperladders. Notice that $n' = n$ implies that $\psi = 1_{\underline{J}(n)}$ is unique. Hence, $\Psi_{1,\boldsymbol{X},\boldsymbol{Y}}$ as well as the functor $\underline{R}^1_{nn} = 1_{\underline{\mathcal{M}}_n}$ is the unique identity functor. Finally, if $n \leq n' \leq n''$ then, for every pair ψ, ψ' as above, the functors $\underline{R}^{\psi}_{nn'} : \underline{\mathcal{M}}_{n'} \to \underline{\mathcal{M}}_n$, $\underline{R}^{\psi'}_{n'n''} : \underline{\mathcal{M}}_{n''} \to \underline{\mathcal{M}}_{n'}$ and $\underline{R}^{\psi'\psi}_{nn''} : \underline{\mathcal{M}}_{n''} \to \underline{\mathcal{M}}_n$ satisfy $\underline{R}^{\psi}_{nn'}\underline{R}^{\psi'}_{n'n''} = \underline{R}^{\psi'\psi}_{nn''}$. The last statement is now obvious. □

Let X and Y be compact metric spaces, and let $\boldsymbol{p} : X \to \boldsymbol{X}$ and $\boldsymbol{q} : Y \to \boldsymbol{Y}$ be associated (limits) in $cANR$ or $cPol$ with X and Y respectively. Let $\boldsymbol{p}' : X \to \boldsymbol{X}'$ and $\boldsymbol{q}' : Y \to \boldsymbol{Y}'$ be an other such a pair. Then there is a (unique) pair of natural isomorphisms $\boldsymbol{u} : \boldsymbol{X} \to \boldsymbol{X}'$, $\boldsymbol{v} : \boldsymbol{Y} \to \boldsymbol{Y}'$ in the pro-category tow-$HcANR$ such that $\boldsymbol{u}H(\boldsymbol{p}) = H(\boldsymbol{p}')$ and $\boldsymbol{v}H(\boldsymbol{q}) = H(\boldsymbol{q}')$, where H is the homotopy functor. Let us choose a pair of representatives $(u', u'_s) : \boldsymbol{X}' \to \boldsymbol{X}$, $(v, v_j) : \boldsymbol{Y} \to \boldsymbol{Y}'$ of $\boldsymbol{u}^{-1}$ and $\boldsymbol{v}$ in inv-$cANR$ respectively, such that the index functions u' and v are increasing and unbounded and $u'v \geq 1_{\mathbb{N}}$. Let $n \in \mathbb{N}$ and let $F_n = (f_{\boldsymbol{j}^n}) : \boldsymbol{X} \to \boldsymbol{Y}$ be a free n-hyperladder. We want to show that (u', u'_s), (v, v_j) and F_n yield a free n-hyperladder $F'_n = (f'_{\boldsymbol{j}^n}) : \boldsymbol{X}' \to \boldsymbol{Y}'$ obtained by "composing" (u', u'_s), F_n and (v, v_j).

$$\begin{array}{ccc}
\boldsymbol{X} & \overset{(u',u'_i)}{\longleftarrow} & \boldsymbol{X}' \\
F_n \downarrow & & \downarrow F'_n \\
\boldsymbol{Y} & \underset{(v,v_j)}{\longrightarrow} & \boldsymbol{Y}'
\end{array}$$

Even more, if $F_n \simeq G_n$ then $F'_n \simeq G'_n$. Let us first consider the simplest case $n = 1$. Let $\boldsymbol{j}^1 = (j_1, j_2) \in \boldsymbol{J}(1)$. Put $t_1 = v(j_1)$ and $t_2 = v(j_2)$. Then $\boldsymbol{t}^1 = (t_1, t_2) \in \boldsymbol{J}(1)$, and consider the corresponding 1-ladder $f_{\boldsymbol{t}^1} \in F_1$ of $\boldsymbol{X}$ to $\boldsymbol{Y}$. We now define the 1-ladder $f'_{\boldsymbol{j}^1} : \boldsymbol{X}' \to \boldsymbol{Y}'$ by means of $f_{\boldsymbol{t}^1}$ and the restrictions of (u', u'_s) to the appropriate subset of $[s_1, s_2]_{\mathbb{N}} = [t_1, t_2]_{\mathbb{N}}$ and of (v, v_j) to the appropriate subset of $[j_1, j_2]_{\mathbb{N}}$. More precisely, the index function of $f'_{\boldsymbol{j}^1}$ is defined by means of the composition $u'fv$, where f is the index function of $f_{\boldsymbol{t}^1}$. Notice that $f'(j_1) = u'fv(j_1) \geq i_1 = j_1$ holds by our assumptions and by $f(t_1) \geq s_1 = t_1$. Obviously, such an $f'_{\boldsymbol{j}^1}$ is empty if and only if $f'(j_1) \geq i_2 = j_2$. Let $F'_1 = (f'_{\boldsymbol{j}^1})$, $\boldsymbol{j}^1 \in \boldsymbol{J}(1)$, be the family of all such 1-ladders. Then, clearly, $F'_1 : \boldsymbol{X}' \to \boldsymbol{Y}'$ is a free 1-hyperladder. Let a free 1-hyperladder $G_1 = (g_{\boldsymbol{t}^1}) : \boldsymbol{X} \to \boldsymbol{Y}$ be given such that $G_1 = (g_{\boldsymbol{t}^1}) \simeq (f_{\boldsymbol{t}^1}) = F_1$, and let $G'_1 = (g'_{\boldsymbol{j}^1}) : \boldsymbol{X}' \to \boldsymbol{Y}'$ be constructed in the same way by means of (u', u'_s), $(g_{\boldsymbol{t}^1})$ and (v, v_j), i.e. $(g'_{\boldsymbol{j}^1}) =$ "$(v, v_j)(g_{\boldsymbol{t}^1})(u', u_s)$". Then a straightforward verification shows that $G'_1 = (g'_{\boldsymbol{j}^1}) \simeq (f'_{\boldsymbol{j}^1}) = F'_1$. Namely, one has to choose an existing index s^1 for $t_1 = v(j_1)$, and then a desired index $i^1 \geq u'(s^1) \geq s^1$. Then, for every $j_2 > i^1$ and $t_2 = v(j_2) \geq i^1$, the both relations $f_{\boldsymbol{t}^1} \simeq g_{\boldsymbol{t}^1}$ and $f'_{\boldsymbol{j}^1} \simeq g'_{\boldsymbol{j}^1}$ hold. If $n > 1$, one applies the same construction on every λ-block, inductively on $\lambda = 1, \dots, n$. Therefore, for every $n \in \mathbb{N}$, there exists a correspondence between the morphisms sets

$$h = h^{(u', u'_s)}_{(v, v_j)} : \underline{\mathcal{A}}_n(\boldsymbol{X}, \boldsymbol{Y}) \to \underline{\mathcal{A}}_n(\boldsymbol{X}', \boldsymbol{Y}'), \quad h(F_n) = F'_n,$$

where (shortly writing) $F'_n =$ "$(u', u'_s)F_n(v, v_j)$". Moreover, there is a correspondence between the quotient sets

$$h = h^{\boldsymbol{u}^{-1}}_{\boldsymbol{v}} : \underline{\mathcal{A}}_n(\boldsymbol{X}, \boldsymbol{Y})/\simeq \to \underline{\mathcal{A}}_n(\boldsymbol{X}', \boldsymbol{Y}')/\simeq, \quad h([F_n]) = [F'_n],$$

where (shortly writing) $[F'_n] =$"$\boldsymbol{v}[F_n]\boldsymbol{u}^{-1}$". Let us observe that by the same construction (using the appropriate representatives (u, u_i) and (v', v'_t) of $\boldsymbol{u}$ and $\boldsymbol{v}^{-1}$ respectively), every $F'_n : \boldsymbol{X}' \to \boldsymbol{Y}'$ gives an $F_n : \boldsymbol{X} \to \boldsymbol{Y}$ preserving the homotopy relation. More precisely, every free n-hyperladder $F'_n : \boldsymbol{X}' \to \boldsymbol{Y}'$ (homotopy class $[F'_n]$) yields a free n-hyperladder $F_n : \boldsymbol{X} \to \boldsymbol{Y}$ (homotopy class $[F_n]$), $F_n =$"$(v', v'_t)F'_n(u, u_i)$", ($[F_n] =$"$\boldsymbol{v}^{-1}[F'_n]\boldsymbol{u}$").
It is readily seen that $h^{\boldsymbol{u}^{-1}}_{\boldsymbol{v}}$ is a bijection, and that $h^{\boldsymbol{u}^{-1}}_{\boldsymbol{u}}$ preserves the identity. However, one can not well define the value of such a bijection on a composite, because the composition of homotopy classes can not be well defined (Remark 1.).

According to the above consideration, we may state the following theorem:

Theorem 2. *Let X and Y be compact metric spaces, and let $\boldsymbol{p} : X \to \boldsymbol{X}$, $\boldsymbol{p}' : X \to \boldsymbol{X}'$, $\boldsymbol{q} : Y \to \boldsymbol{Y}$ and $\boldsymbol{q}' : Y \to \boldsymbol{Y}'$ be inverse limits in tow-cANR. Then, for every $n \in \mathbb{N}$, there exists a bijection*

$$h \equiv h^{\boldsymbol{u}^{-1}}_{\boldsymbol{v}} : \underline{\mathcal{A}}_n(\boldsymbol{X}, \boldsymbol{Y})/\simeq \to \underline{\mathcal{A}}_n(\boldsymbol{X}', \boldsymbol{Y}')/\simeq, \quad h[F_n] = \text{"}\boldsymbol{v}[F_n]\boldsymbol{u}^{-1}\text{"},$$

where $\boldsymbol{u} : \boldsymbol{X} \to \boldsymbol{X}'$ and $\boldsymbol{v} : \boldsymbol{Y} \to \boldsymbol{Y}'$ are the unique isomorphisms in the pro-category tow-HcANR satisfying $\boldsymbol{u}H(\boldsymbol{p}) = H\boldsymbol{p}'$ and $\boldsymbol{v}H(\boldsymbol{q}) = H\boldsymbol{q}'$. If $\boldsymbol{v} = \boldsymbol{u}$, then $h^{\boldsymbol{u}^{-1}}_{\boldsymbol{u}}([1_{\boldsymbol{X}^n}]) = [1_{\boldsymbol{X}'^n}]$.

The next definition brings a useful notion (at least for a brief writing).

Definition 4. *Let $n \in \mathbb{N}$ and let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be inverse sequences in $c\mathcal{M}$. Then $\boldsymbol{X}$ is said to be n-**alike** $\boldsymbol{Y}$, denoted by $\boldsymbol{X} \overset{n}{\leftrightarrow} \boldsymbol{Y}$, provided there exists a pair of free n-hyperladders $F_n \in \underline{\mathcal{M}}_n(\boldsymbol{X}, \boldsymbol{Y})$, $G_n \in \underline{\mathcal{M}}_n(\boldsymbol{Y}, \boldsymbol{X})$ such that $G_nF_n \simeq 1_{\boldsymbol{X}^n}$ and $F_nG_n \simeq 1_{\boldsymbol{Y}^n}$. If X and Y are compacta, then $X \overset{n}{\leftrightarrow} Y$ is defined by means of $\boldsymbol{X} \overset{n}{\leftrightarrow} \boldsymbol{Y}$ in $\underline{\mathcal{A}}_n$, for any choice of the associated $\boldsymbol{X}$ and $\boldsymbol{Y}$ in cANR or cPol. Further, $\boldsymbol{X}$ is said to be **alike** $\boldsymbol{Y}$ (X is said to be **alike** Y), denoted by $\boldsymbol{X} \leftrightarrow \boldsymbol{Y}$ ($X \leftrightarrow Y$), provided $\boldsymbol{X} \overset{n}{\leftrightarrow} \boldsymbol{Y}$ ($X \overset{n}{\leftrightarrow} Y$) for every $n \in \mathbb{N}$.*

It is obvious that the relations $\overset{n}{\leftrightarrow}$, $n \in \mathbb{N}$, are reflexive and symmetric. We do not know whether they are transitive (compare Corollary 1. below). However, each of them generates an equivalence relation on the object class. On the other hand, the both relations $\leftrightarrow$ are equivalence relations (see Corollary 2. below). In the next section we shall relate $\overset{m}{\leftrightarrow}$ to the S_n- and S^+_n-equivalence and vice versa.

4. The S_n- and S_n^+-equivalence in the category $\underline{\mathcal{A}}_m$

According to Lemma 4.1. of [8], $S(\boldsymbol{X}) = S(\boldsymbol{Y})$ is equivalent to the following two conditions:

(1) For every $m \in \mathbb{N}$, there exists a pair $(f^m, (F_j^m)_{j\in\mathbb{N}})$ consisting of a strictly increasing function $f^m : \mathbb{N} \to \mathbb{N}$ and, for every $j \in \mathbb{N}$, of a family F_j^m of mappings $f_{\alpha j}^m : X_{f^m(j)} \to Y_j$, $\alpha \in A_j^m$, such that

(i) $(\forall j \in \mathbb{N})\ f^m(j) \geq j$;

(ii) $(\forall j_1 < \cdots < j_m \text{ in } \mathbb{N})(\forall \lambda \in [1, m]_{\mathbb{N}})(\exists f_{j_\lambda}^m \in F_{j_\lambda}^m)$
$\lambda \leq \lambda' \Rightarrow f_{j_\lambda}^m p_{f^m(j_\lambda) f^m(j_{\lambda'})} \simeq q_{j_\lambda j_{\lambda'}} f_{j_{\lambda'}}^m$;

(2) For every $m > 1$ there exists a pair $(g^m, (G_i^m)_{i\in\mathbb{N}})$ having properties (i)$'$ and (ii)$'$ analogue to (i) and (ii) respectively, where $g^m : \mathbb{N} \to \mathbb{N}$ is increasing, $g_{\beta i}^m : Y_{g^m(i)} \to X_i$ is a mapping, $\beta \in B_i^m$, and

(iii) $(\forall j_1)(\forall i_1 \geq f^m(j_1))(\forall j_2 \geq g^m(i_1)) \cdots (\forall j_m \geq f^m(j_{m-1}))$
there exist mappings $f_{j_1}^m \in F_{j_1}^m, \ldots, f_{j_n}^m \in F_{j_n}^m$, $g_{i_1}^m \in G_{i_1}^m, \ldots, g_{i_{n-1}}^m \in G_{i_{n-1}}^m$ such that the corresponding diagram

$$\begin{array}{ccccccccc}
X_{f^m(j_1)} & \leftarrow & X_{i_1} & \leftarrow & X_{f^m(j_2)} & \leftarrow \cdots \leftarrow & X_{i_{n-1}} & \leftarrow & X_{f^m(j_n)} \\
\downarrow f_{j_1}^m & & \uparrow g_{i_1}^m & & \downarrow f_{j_2}^m & & \uparrow g_{i_{n-1}}^m & & \downarrow f_{j_n}^m \\
Y_{j_1} & \leftarrow & Y_{g^m(i_1)} & \leftarrow & Y_{j_2} & \leftarrow \cdots \leftarrow & Y_{g^{m-1}(i_{n-1})} & \leftarrow & Y_{j_n}
\end{array}$$

commutes up to homotopy.

Further, we may assume that $f^m, g^m \geq 1_{\mathbb{N}}$. Clearly, for a fixed $m = 2n + 1$, the conditions from above characterize condition $S_n(\boldsymbol{X}, \boldsymbol{Y})$, $n \in \{0\} \cup \mathbb{N}$. In a quite similar way one can characterize condition $S_n^+(\boldsymbol{X}, \boldsymbol{Y})$.

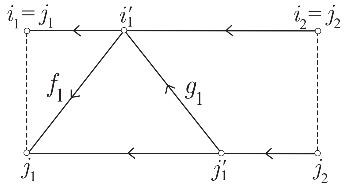
Our first goal is to describe the S_n- and S_n^+-equivalence in terms of the category $\underline{\mathcal{M}}_m$ ($\underline{\mathcal{A}}_m$), i.e. by the m-alikeness, for some (possible maximal) $m \in \mathbb{N}$. Since the S_0-domination and S_0-equivalence are trivial, we are beginning with the S_0^+-domination. More general, in the case of the S_n^+-domination the following fact holds:

Lemma 1. *Let $n \in \{0\} \cup \mathbb{N}$ and let $S_n^+(\boldsymbol{Y}) \leq S_n^+(\boldsymbol{X})$. Then there exist an $F_{n+1} \in \underline{\boldsymbol{L}}_{n+1}(\boldsymbol{X}, \boldsymbol{Y})$ and a $G_{n+1} \in \underline{\boldsymbol{L}}_{n+1}(\boldsymbol{Y}, \boldsymbol{X})$ such that $F_{n+1}G_{n+1} \simeq 1_{\boldsymbol{Y}n+1}$ in $\underline{\mathcal{M}}_{n+1}$.*

Proof. Let $S_n^+(\boldsymbol{Y}) \leq S_n^+(\boldsymbol{X})$, i.e. let condition $S_n^+(\boldsymbol{X}, \boldsymbol{Y})$ be fulfilled, $n \in \{0\} \cup \mathbb{N}$. Then condition (D_{2n+2}) gives rise of the appropriate homotopy commutative diagrams. First, consider the simplest case $n = 0$. A typical diagram corresponding to (D_2) is given below.

$$\begin{tikzcd}
X_{s_1} \arrow[d, "f_1"'] & X_{s_1'} \arrow[l] \arrow[d, "g_1"] \\
Y_{t_1} & Y_{t_1'} \arrow[l]
\end{tikzcd}$$

Namely, condition (D_2) implies that, for every t_1 there exists an s_1, and for every $s_1' \geq s_1$ there exists a $t_1' \geq t_1$, and there exist appropriate mappings f_1 and g_1. Thus, given an ordered pair $j_1 < j_2$, i.e. a $\boldsymbol{j}^1 \in \boldsymbol{J}(1)$, there exists a nonempty 1-ladder $f_{\boldsymbol{j}^1} : \boldsymbol{X} \to \boldsymbol{Y}$ consisting of an existing mapping $f_1 : X_{i_1'} \to Y_{j_1}$ $(t_1 = j_1,\ s_1 = i_1')$ whenever $i_1' < i_2 = j_2$. Also, there exists a nonempty 1-ladder $g_{\boldsymbol{i}^1} : \boldsymbol{Y} \to \boldsymbol{X}$, $\boldsymbol{i}^1 = \boldsymbol{j}^1$, consisting of mappings $g_i = g_1 p_{ii_1'} : X_i \to Y_{j_1'}$, $i = i_1, \dots, i_1'$, $(s_1' = s_1 = i_1',\ t_1' = j_1')$, whenever $i_1' < i_2 = j_2$ and $j_1' < j_2$, where a mapping $g_1 : Y_{j_1'} \to X_{i_1'}$ exists by the assumption.

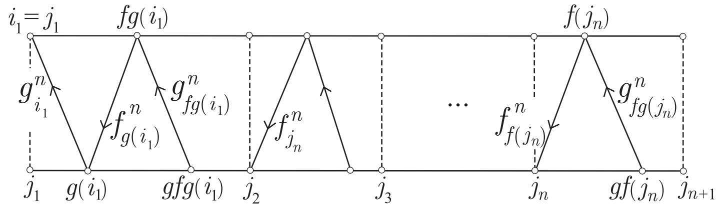


Otherwise, let those 1-ladders be empty. Let $F_1 = (f_{\boldsymbol{j}^1})$ and $G_1 = (g_{\boldsymbol{i}^1})$, $\boldsymbol{j}^1, \boldsymbol{i}^1 \in \boldsymbol{J}(1)$, be the families obtained by all such 1-ladders. Then, clearly, $F_1 : \boldsymbol{X} \to \boldsymbol{Y}$ and $G_1 : \boldsymbol{Y} \to \boldsymbol{X}$ are free 1-hyperladders, i.e. $F_1 \in \boldsymbol{L}_1(\boldsymbol{X}, \boldsymbol{Y})$ and $G_1 \in \boldsymbol{L}_1(\boldsymbol{Y}, \boldsymbol{X})$. By construction, given a j_1, one may put $j^1 = j_1'$. Then, for every $j_2 > j^1$, the homotopy condition for $F_1 G_1$ and $1_{\boldsymbol{Y}1}$ obviously holds, i.e. $F_1 G_1 \simeq 1_{\boldsymbol{Y}1}$ in $\underline{\mathcal{M}}_1(\boldsymbol{Y}, \boldsymbol{Y})$.

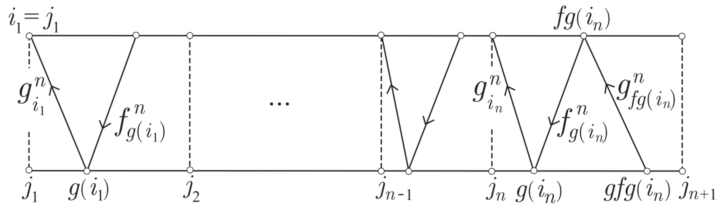
Let us now consider the general case. Let $S_n^+(\boldsymbol{Y}) \leq S_n^+(\boldsymbol{X})$, i.e. let condition $S_n^+(\boldsymbol{X}, \boldsymbol{Y})$ be fulfilled, $n > 0$. Then the diagrams corresponding to condition (D_{2n+2}) give rise of mappings to obtain a pair of free $(n+1)$-hyperladders $F_{n+1} \in \boldsymbol{L}_{n+1}(\boldsymbol{X}, \boldsymbol{Y})$, $G_{n+1} \in \boldsymbol{L}_{n+1}(\boldsymbol{Y}, \boldsymbol{X})$. Indeed, given a $\boldsymbol{j}^{n+1} \in \boldsymbol{J}(n+1)$, one can construct the $(n+1)$-ladder $f_{\boldsymbol{j}^{n+1}} : \boldsymbol{X} \to \boldsymbol{Y}$ by means of the mappings $f_\lambda : X_{i_\lambda'} \to Y_{j_\lambda}$ $(t_\lambda = j_\lambda,\ s_\lambda = i_\lambda')$ whenever $i_\lambda' < i_{\lambda+1} = j_{\lambda+1}$, $\lambda = 1, \dots, n+1$. Otherwise, let the λ-block of $f_{\boldsymbol{j}^{n+1}}$ be empty. Also, let the $(n+1)$-ladder $g_{\boldsymbol{i}^{n+1}} : \boldsymbol{Y} \to \boldsymbol{X}$, $\boldsymbol{i}^{n+1} = \boldsymbol{j}^{n+1}$, consist of the mappings $g_\lambda : Y_{j_\lambda'} \to X_{i_1'}$ $(s_\lambda' = s_\lambda = i_1',\ t_\lambda' = j_\lambda')$, whenever $i_\lambda' < i_{\lambda+1} = j_{\lambda+1}$ and $j_\lambda' < j_{\lambda+1}$, $\lambda = 1, \dots, n+1$. Otherwise, let the λ-block of $g_{\boldsymbol{i}^{n+1}}$ be empty. (One can also use the characterization from above: The nonempty λ-blocks are those that satisfy $f^n(j_\lambda) < j_{\lambda+1}$ - for $f_{\boldsymbol{j}^{n+1}}$, and $g^n f^n(j_\lambda) < j_{\lambda+1}$ - for $g_{\boldsymbol{i}^{n+1}}$, $\boldsymbol{i}^{n+1} = \boldsymbol{j}^{n+1}$.) Now, a straightforward verification shows that $F_{n+1} G_{n+1} \simeq 1_{\boldsymbol{Y}n+1}$ in $\underline{\mathcal{M}}_{n+1}(\boldsymbol{Y}, \boldsymbol{Y})$. Namely, given an $m \leq n+1$, for every j_1 put $j^1 = j_1'$, for every $j_2 > j^1$ put $j^2 = j_2'$, ... , for every $j_m > j^{m-1}$ put $j^m \geq j_m'$. And then, for every choice of $j_{n+2} > \cdots > j_{m+1} > j^m$, the corresponding $(n+1)$-ladders $f_{\boldsymbol{j}^{n+1}} g_{\boldsymbol{j}^{n+1}} \in F_{n+1} G_{n+1}$ and $1_{\boldsymbol{Y}\boldsymbol{j}^{n+1}} \in 1_{\boldsymbol{Y}n+1}$ are m-homotopic, $f_{\boldsymbol{j}^{n+1}} g_{\boldsymbol{j}^{n+1}} \simeq_m 1_{\boldsymbol{Y}\boldsymbol{j}^{n+1}}$. □

Lemma 2. *Let $n \in \mathbb{N}$ and $S_n(\boldsymbol{Y}) \leq S_n(\boldsymbol{X})$. Then there exist $F_n, F_n' \in \underline{\boldsymbol{L}}_n(\boldsymbol{X}, \boldsymbol{Y})$ and $G_n, G_n' \in \underline{\boldsymbol{L}}_n(\boldsymbol{Y}, \boldsymbol{X})$ such that $F_n G_n \simeq 1_{\boldsymbol{Y}n}$ and $G_n' F_n' \simeq 1_{\boldsymbol{X}n}$ in $\underline{\mathcal{M}}_n$.*

Proof. Let $S_n(\boldsymbol{Y}) \leq S_n(\boldsymbol{X})$, i.e. let condition $S_n(\boldsymbol{Y}, \boldsymbol{X})$ be fulfilled, $n \in \mathbb{N}$. Then the corresponding condition is (D_{2n+1}). We are applying the characterization from above. Denote the appropriate index functions g^n and f^n by g and f respectively. Let an $\boldsymbol{i}^n \in \boldsymbol{J}(n)$ be given. We define the first blocks of $g_{\boldsymbol{i}^n}$ and $f_{\boldsymbol{j}^n}$, $\boldsymbol{j}^n = \boldsymbol{i}^n$, by means of mappings $g_{i_1}^n, g_{fg(i_1)}^n$ and $f_{g(i_1)}^n$, and the bonding mappings, respectively, whenever $gfg(i_1) < j_2 = i_2$; otherwise, let their first blocks be empty. For every other $\lambda = 2, \dots, n$, we define the λ-blocks of $g_{\boldsymbol{i}^n}$ and $f_{\boldsymbol{j}^n}$ by means of mappings $g^n_{f(j_\lambda)}$ and $f^n_{j_\lambda}$, and the bonding mappings, respectively, whenever $gf(j_\lambda) < j_{\lambda+1} = i_{\lambda+1}$; otherwise, let their λ-blocks be empty. This construction yields a $G_n \in \underline{\boldsymbol{L}}_n(\boldsymbol{Y}, \boldsymbol{X})$ and an $F_n \in \underline{\boldsymbol{L}}_n(\boldsymbol{X}, \boldsymbol{Y})$ such that $F_n G_n \simeq 1_{\boldsymbol{Y}^n}$ in $\underline{\mathcal{M}}_n$.



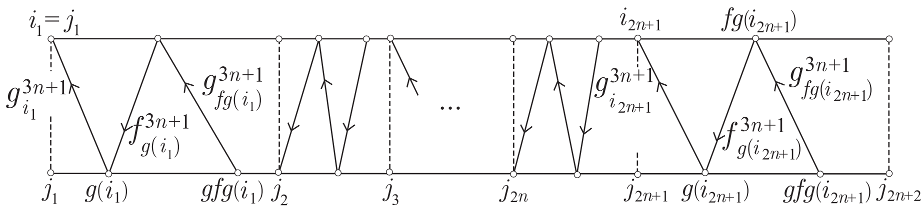
On the other hand, for every $\lambda = 1, \dots, n-1$, we can define the λ-blocks of $g'_{\boldsymbol{i}^n}$ and $f'_{\boldsymbol{j}^n}$, $\boldsymbol{j}^n = \boldsymbol{i}^n$, by means of mappings $g^n_{i_\lambda}$ and $f^n_{g(i_\lambda)}$, and the bonding mappings, respectively, whenever $fg(i_\lambda) < i_{\lambda+1} = j_{\lambda+1}$; otherwise, let these blocks be empty. For $\lambda = n$, we define the n-blocks of $g_{\boldsymbol{i}^n}$ and $f_{\boldsymbol{j}^n}$ by means of mappings $g^n_{i_n}$, $g^n_{fg(i_n)}$ and $f^n_{g(i_n)}$, and the bonding mappings, respectively, whenever $gfg(i_n) < j_{n+1} = i_{n+1}$; otherwise, let their n-blocks be empty.



The construction yields a $G'_n \in \underline{\boldsymbol{L}}_n(\boldsymbol{Y}, \boldsymbol{X})$ and an $F'_n \in \underline{\boldsymbol{L}}_n(\boldsymbol{X}, \boldsymbol{Y})$ such that $G'_n F'_n \simeq 1_{\boldsymbol{X}^n}$ in $\mathcal{M}_n$. $\square$

Lemma 3. *If* $S_{3n+1}(\boldsymbol{Y}) \le S_{3n+1}(\boldsymbol{X})$ *(or* $S_{3n+1}(\boldsymbol{X}) \le S_{3n+1}(\boldsymbol{Y})$*),* $n \in \{0\} \cup \mathbb{N}$*, then* $\boldsymbol{X} \overset{2n+1}{\leftrightarrow} \boldsymbol{Y}$.

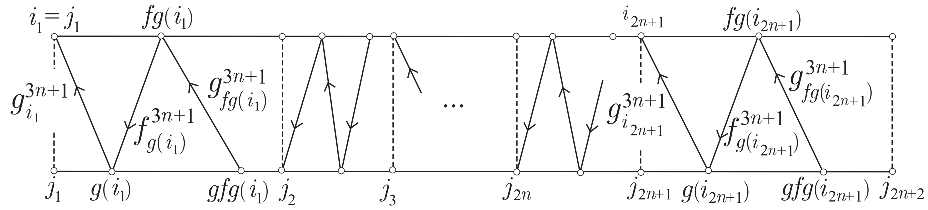
Proof. Let $S_{3n+1}(\boldsymbol{Y}) \le S_{3n+1}(\boldsymbol{X})$, i.e. let condition $S_{3n+1}(\boldsymbol{Y}, \boldsymbol{X})$ be fulfilled, $n \in \{0\} \cup \mathbb{N}$. Recall that the corresponding condition is (D_{6n+3}). As before, denote the appropriate index functions g^{3n+1} and f^{3n+1} by g and f respectively. Consider a λ-block of an $\boldsymbol{i}^{2n+1} \in \boldsymbol{J}(2n+1)$, $\lambda \in 1, \dots, 2n+1$. If λ is odd and $gfg(i_\lambda) < j_{\lambda+1} = i_{\lambda+1}$, let $g_{\boldsymbol{i}^{2n+1}}$ and $f_{\boldsymbol{j}^{2n+1}}$, $\boldsymbol{j}^{2n+1} = \boldsymbol{i}^{2n+1}$, on these blocks be defined by means of the existing mappings $g^{3n+1}_{i_\lambda}, g^{3n+1}_{fg(i_\lambda)}$ and $f^{3n+1}_{g(i_\lambda)}$, and the bonding mappings, respectively; otherwise, let their odd λ-blocks be empty. If λ is even and $fgf(j_\lambda) < i_{\lambda+1} = j_{\lambda+1}$, let $g_{\boldsymbol{i}^{2n+1}}$ and $f_{\boldsymbol{j}^{2n+1}}$ on these blocks be defined by means of the existing mappings $g^{3n+1}_{f(j_\lambda)}$ and $f^{3n+1}_{j_\lambda}, f^{3n+1}_{gf(j_\lambda)}$, and the bonding mappings, respectively; otherwise, let their even λ-blocks be empty.



This construction yields a pair of free $(2n+1)$-hyperladders $G_{2n+1} \in \underline{\boldsymbol{L}}_{2n+1}(\boldsymbol{Y}, \boldsymbol{X})$, $F_{2n+1} \in \underline{\boldsymbol{L}}_{2n+1}(\boldsymbol{X}, \boldsymbol{Y})$. It is readily seen that $F_{2n+1}G_{2n+1} \simeq 1_{\boldsymbol{Y}2n+1}$ and $G_{2n+1}F_{2n+1} \simeq 1_{\boldsymbol{X}2n+2}$ in $\underline{\mathcal{M}}_{2n+1}$ hold. It means that $\boldsymbol{X} \overset{2n+1}{\leftrightarrow} \boldsymbol{Y}$. The proof in the case $S_{3n+1}(\boldsymbol{X}) \leq S_{3n+1}(\boldsymbol{Y})$ is quite similar. □

Lemma 4. *If $S^+_{3n+2}(\boldsymbol{Y}) \leq S^+_{3n+2}(\boldsymbol{X})$ (or $S^+_{3n+2}(\boldsymbol{X}) \leq S^+_{3n+2}(\boldsymbol{Y})$), $n \in \{0\} \cup \mathbb{N}$, then $\boldsymbol{X} \overset{2n+2}{\leftrightarrow} \boldsymbol{Y}$.*

Proof. Let $S^+_{3n+2}(\boldsymbol{Y}) \leq S^+_{3n+2}(\boldsymbol{X})$, i.e. let condition $S^+_{3n+2}(\boldsymbol{X}, \boldsymbol{Y})$ be fulfilled, $n \in \{0\} \cup \mathbb{N}$. Recall that the corresponding condition is (D_{6n+6}). Let us apply the above characterization. Denote the appropriate index functions f^{3n+2} and g^{3n+2} by f and g respectively. Consider a λ-block of a $\boldsymbol{j}^{2n+2} \in \boldsymbol{J}(2n+2)$, $\lambda \in 1, \ldots, 2n+2$. If λ is odd and $fgf(j_\lambda) < i_{\lambda+1} = j_{\lambda+1}$, let $f_{\boldsymbol{j}^{2n+2}}$ and $g_{\boldsymbol{i}^{2n+2}}$, $\boldsymbol{i}^{2n+2} = \boldsymbol{j}^{2n+2}$, on these blocks be defined by means of the existing mappings $f^{3n+2}_{j_\lambda}$, $f^{3n+2}_{gf(j_\lambda)}$ and $g^{3n+2}_{f(j_\lambda)}$, and the bonding mappings, respectively; otherwise, let their odd λ-blocks be empty. If λ is even and $gfg(i_\lambda) < j_{\lambda+1} = i_{\lambda+1}$, let $f_{\boldsymbol{j}^{2n+2}}$ and $g_{\boldsymbol{i}^{2n+2}}$ on these blocks be defined by means of the existing mappings $f^{3n+2}_{g(i_\lambda)}$ and $g^{3n+2}_{i_\lambda}$, $g^{3n+2}_{fg(i_\lambda)}$, and the bonding mappings, respectively; otherwise, let their even λ-blocks be empty.



In this way a pair of free $(2n+2)$-hyperladders $F_{2n+2} \in \underline{\boldsymbol{L}}_{2n+2}(\boldsymbol{X}, \boldsymbol{Y})$, $G_{2n+2} \in \underline{\boldsymbol{L}}_{2n+2}(\boldsymbol{Y}, \boldsymbol{X})$ is constructed. It is readily seen that $G_{2n+2}F_{2n+2} \simeq 1_{\boldsymbol{X}2n+2}$ and $F_{2n+2}G_{2n+2} \simeq 1_{\boldsymbol{Y}2n+2}$ in $\underline{\mathcal{M}}_{2n+2}$ hold. Thus, $\boldsymbol{X} \overset{2n+2}{\leftrightarrow} \boldsymbol{Y}$. The proof in the case $S^+_{3n+2}(\boldsymbol{X}) \leq S^+_{3n+2}(\boldsymbol{Y})$ is quite similar. □

Remark 2. *It seems that, in general circumstances, the estimation integers $m \in \mathbb{N}$ of $\underline{\mathcal{M}}_m$ in Lemmata 1. - 4. are the best possible (maximal).*

The next theorems are the obvious consequences of the above lemmata.

Theorem 3. *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be compact ANR inverse sequences associated with metric compacta X and Y respectively, and let $n \in \{0\} \cup \mathbb{N}$.*

(i) If $S_n(Y) \leq S_n(X)$ and $n > 0$, then there exist $F_n, F'_n \in \underline{\mathcal{A}}_n(\boldsymbol{X}, \boldsymbol{Y})$ and $G_n, G'_n \in \underline{\mathcal{A}}_n(\boldsymbol{Y}, \boldsymbol{X})$ such that $F_nG_n \simeq 1_{\boldsymbol{Y}n}$ and $G'_nF'_n \simeq 1_{\boldsymbol{X}n}$ in $\underline{\mathcal{A}}_n$.

(ii) If $S_{3n+1}(Y) = S_{3n+1}(X)$, then $X \overset{2n+1}{\leftrightarrow} Y$.

Proof. The proof follows immediately by Lemmata 2. and 3. □

Theorem 4. *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be compact ANR inverse sequences associated with metric compacta X and Y respectively, and let $n \in \{0\} \cup \mathbb{N}$.*

(i) If $S^+_n(Y) \leq S^+_n(X)$, then there exist an $F_{n+1} \in \underline{\mathcal{A}}_{n+1}(\boldsymbol{X}, \boldsymbol{Y})$ and a $G_{n+1} \in \underline{\mathcal{A}}_{n+1}(\boldsymbol{Y}, \boldsymbol{X})$ such that $F_{n+1}G_{n+1} \simeq 1_{\boldsymbol{Y}n+1}$ in $\underline{\mathcal{A}}_{n+1}$.

(ii) If $S^+_{3n+2}(Y) = S^+_{3n+2}(X)$, then $X \overset{2n+2}{\leftrightarrow} Y$.

Proof. The proof follows immediately by Lemmata 1. and 4. $\square$

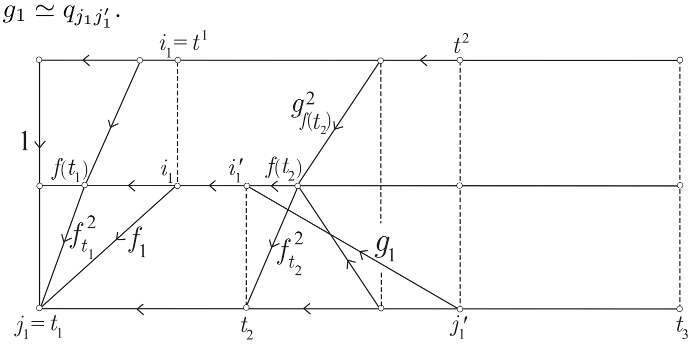
Let us now consider the converse problem: How do the appropriate homotopy relations (m-alikeness) in the category $\underline{\mathcal{A}}_m$, $m \in \mathbb{N}$, imply the S_n- and S_n^+-domination (the S_n- and S_n^+-equivalence) of compacta?

Lemma 5. *If there exist an $F_n \in \underline{\mathcal{M}}_n(\boldsymbol{X}, \boldsymbol{Y})$ and a $G_n \in \underline{\mathcal{M}}_n(\boldsymbol{Y}, \boldsymbol{X})$, $n > 1$, such that $F_n G_n \simeq 1_{\boldsymbol{Y}^n}$, then $S_0^+(\boldsymbol{Y}) \le S_0^+(\boldsymbol{X})$.*

Proof. According to Theorem 1., it suffices to prove that $F_2 G_2 \simeq 1_{\boldsymbol{Y}^2}$ implies $S_0^+(\boldsymbol{Y}) \le S_0^+(\boldsymbol{X})$. By our assumption (see Definition 3.), for $m = n = 2$,

$$(\forall t_1)(\exists t^1 \ge t_1)(\forall t_2 > t^1)(\exists t^2 \ge t_2)(\forall t_3 > t^2) \quad f_{\boldsymbol{t}^2} g_{\boldsymbol{t}^2} \simeq 1_{\boldsymbol{Y}\boldsymbol{t}^2}$$

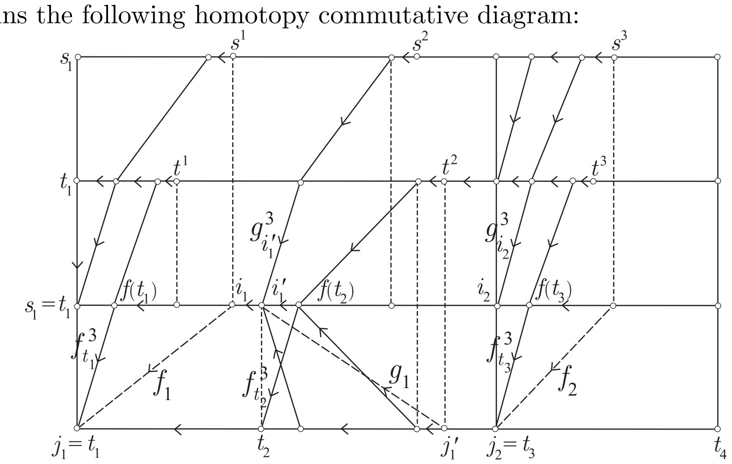
holds. Now, given a $j_1 \in \mathbb{N}$, take $t_1 = j_1$ and put $i_1 = t^1$. Let $i_1' \ge i_1$. Take $t_2 = i_1'$ (or any $t_2 > i_1'$ if $t^1 = t_1$ happens), and put $j_1' = t^2$. Let $t_3 > t^2$. Then the corresponding homotopy relation $f_{\boldsymbol{t}^2} g_{\boldsymbol{t}^2} \simeq 1_{\boldsymbol{Y}\boldsymbol{t}^2}$ yields a mapping $f_1 = f_{t_1}^2 p_{f(t_1) i_1} : X_{i_1} \to Y_{t_1} = Y_{j_1}$ and a mapping $g_1 = p_{i_1' f(t_2)} g_{f(t_2)}^2 q_{g f(t_2) j_1'} : Y_{j_1'} = Y_{t^2} \to X_{i_1'}$ such that $f_1 p_{i_1 i_1'} g_1 \simeq q_{j_1 j_1'}$.



Thus, condition $S_0^+(\boldsymbol{X}, \boldsymbol{Y})$ holds, i.e. $S_0^+(\boldsymbol{Y}) \le S_0^+(\boldsymbol{X})$. $\square$

Lemma 6. *If $\boldsymbol{X} \overset{2n+1}{\leftrightarrow} \boldsymbol{Y}$ in $\underline{\mathcal{M}}_{2n+1}$, $n \in \{0\} \cup \mathbb{N}$, then $S_n(\boldsymbol{Y}) = S_n(\boldsymbol{X})$.*

Proof. If $n = 0$, the claim is trivial. Let $n = 1$, and let there exist an $F_3 \in \underline{\mathcal{M}}_3(\boldsymbol{X}, \boldsymbol{Y})$ and a $G_3 \in \underline{\mathcal{M}}_3(\boldsymbol{Y}, \boldsymbol{X})$ such that $G_3 F_3 \simeq 1_{\boldsymbol{X}^3}$ and $F_3 G_3 \simeq 1_{\boldsymbol{Y}^3}$. By the procedure similar to the one used in the proof of Lemma 5. ($m = n = 3$), one obtains the following homotopy commutative diagram:



More precisely,

$$(\forall t_1)(\exists t^1 \geq t_1)(\forall t_2 > t^1)(\exists t^2 \geq t_2)$$
$$(\forall t_3 > t^2)(\exists t^3 \geq t_3)(\forall t_4 > t^3) \quad f_{\boldsymbol{t}^3} g_{\boldsymbol{t}^3} \simeq 1_{\boldsymbol{Y}\boldsymbol{t}^3}$$

and

$$(\forall s_1)(\exists s^1 \geq s_1)(\forall s_2 > s^1)(\exists s^2 \geq s_2)$$
$$(\forall s_3 > s^2)(\exists s^3 \geq s_3)(\forall s_4 > s^3) \quad g_{\boldsymbol{s}^3} f_{\boldsymbol{s}^3} \simeq 1_{\boldsymbol{X}\boldsymbol{s}^3}$$

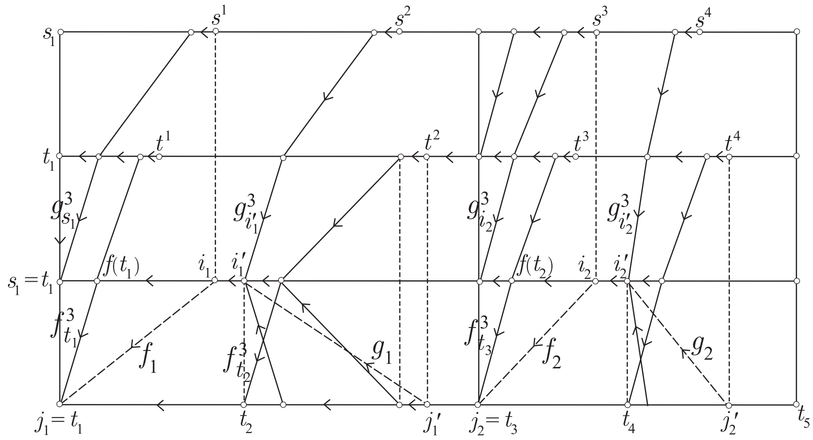
Thus,

$$(\forall j_1)(\exists i_1)(\forall i_1' \geq i_1)(\exists j_1' \geq j_1)(\forall j_2 \geq j_1')(\exists i_2 \geq i_1')$$

and there exist mappings f_1, g_1 and f_2 (dotted arrows - compositions of the appropriate f_s and g_t with the bonding mappings) such that the corresponding diagram commutes up to homotopy. It proves that condition $S_1(\boldsymbol{X},\boldsymbol{Y})$ holds, i.e. $S_1(\boldsymbol{X}) \leq S_1(\boldsymbol{Y})$. Since the assumptions are symmetric, $S_1(\boldsymbol{Y}) \leq S_1(\boldsymbol{X})$ also holds. Thus, $S_1(\boldsymbol{Y}) = S_1(\boldsymbol{X})$. If $n > 1$, the proof works in the same way. □

Lemma 7. *If $\boldsymbol{X} \overset{2n}{\leftrightarrow} \boldsymbol{Y}$ in $\underline{\mathcal{M}}_{2n}$, $n \in \mathbb{N}$, then $S^+_{n-1}(\boldsymbol{Y}) = S^+_{n-1}(\boldsymbol{X})$.*

Proof. If $n = 1$, then by Definition 4. and Lemma 5., $S^+_0(\boldsymbol{Y}) \leq S^+_0(\boldsymbol{X})$ and $S^+_0(\boldsymbol{X}) \leq S^+_0(\boldsymbol{Y})$. (We do not even need a unique pair F_2, G_2!) Hence, $S^+_0(\boldsymbol{Y}) = S^+_0(\boldsymbol{X})$. Let $n > 1$. First, consider the case $n = 2$. By applying the both homotopy relations $F_4G_4 \simeq 1_{\boldsymbol{Y}4}$ and $G_4F_4 \simeq 1_{\boldsymbol{X}4}$ simultaneously (starting with $F_4G_4 \simeq 1_{\boldsymbol{Y}4}$), for $m = n = 4$, the following procedure is possible: Given a j_1, take $t_1 = s_1 = j_1$ and put $i_1 = \max\{t^1, s^1\}$; let $i_1' \geq i_1$; take $t_2 = s_2 = i_1'$, and put $j_1' = \max\{t^2, s^2\}$; let $j_2 \geq j_1'$; take $t_3 = s_3 = j_2$, and put $i_2 = \max\{t^3, s^3\}$; let $i_2' \geq i_2$; take $t_4 = s_4 = i_2'$, and put $j_2' = \max\{t^4, s^4\}$, finally, let $t_5 = s_5 > j_2'$.



Since $f_{\boldsymbol{t}^4} g_{\boldsymbol{t}^4} \simeq 1_{\boldsymbol{Y}\boldsymbol{t}^4}$ and $g_{\boldsymbol{s}^4} f_{\boldsymbol{s}^4} \simeq 1_{\boldsymbol{X}\boldsymbol{s}^4}$, $\boldsymbol{s}^4 = \boldsymbol{t}^4$, the needed mappings (dotted arrows) exist such that condition $S^+_1(\boldsymbol{X},\boldsymbol{Y})$ holds, i.e. $S^+_1(\boldsymbol{Y}) \leq S^+_1(\boldsymbol{X})$. In the same way, beginning with $G_4F_4 \simeq 1_{\boldsymbol{X}4}$, one can prove that $S^+_1(\boldsymbol{X}) \leq S^+_1(\boldsymbol{Y})$ holds. Therefore, $S^+_1(\boldsymbol{Y}) = S^+_1(\boldsymbol{X})$. If $n > 2$, the proof is quite similar. □

The next theorems follow by the above Lemmata 6. and 7. respectively.

Theorem 5. *Let X and Y be metric compacta, and let $n \in \{0\} \cup \mathbb{N}$. If $X \overset{2n+1}{\leftrightarrow} Y$, then $S_n(Y) = S_n(X)$.*

Theorem 6. *Let X and Y be metric compacta, and let $n \in \mathbb{N}$. If $X \overset{2n}{\leftrightarrow} Y$, then $S^+_{n-1}(Y) = S^+_{n-1}(X)$.*

Corollary 1. *Let X, Y and Z be metric compacta, and let $n \in \mathbb{N}$, $n \geq 3$. If $X \overset{n}{\leftrightarrow} Y$ and $Y \overset{n}{\leftrightarrow} Z$, then $X \overset{[\frac{n}{3}]}{\leftrightarrow} Z$.*

Proof. Let $n \in \mathbb{N}$, $n \geq 3$, and let $X \overset{n}{\leftrightarrow} Y$ and $Y \overset{n}{\leftrightarrow} Z$. If n is odd, i.e. $n = 2k+1$, $k \in \mathbb{N}$, then Theorem 5. implies $S_k(Y) = S_k(X)$ and $S_k(Z) = S_k(Y)$. Thus, $S_k(Z) = S_k(X)$. By Theorem 3.(ii), $S_k(Z) = S_k(X)$ implies $X \overset{l}{\leftrightarrow} Z$ whenever $l \leq [\frac{2k+1}{3}] = [\frac{n}{3}]$. If n is even, i.e. $n = 2(k+1)$, $k \in \mathbb{N}$, then Theorem 6. implies $S^+_k(Y) = S^+_k(X)$ and $S^+_k(Z) = S^+_k(Y)$. Thus, $S^+_k(Z) = S^+_k(X)$. By Theorem 4.(ii), $S^+_k(Z) = S^+_k(X)$ implies $X \overset{l}{\leftrightarrow} Z$ whenever $l \leq [\frac{2(k+1)}{3}] = [\frac{n}{3}]$. $\square$

Remark 3. *By Theorem 1., there are functors of $\underline{\mathcal{A}}_{n'}$ to $\underline{\mathcal{A}}_n$, $n \leq n'$, keeping the objects fixed and preserving the homotopy relation. One can easily construct a functor of $\underline{\mathcal{A}}_n$ to $\underline{\mathcal{A}}_{n'}$ keeping the objects fixed. For instance, let $n = 1$ and $n' = 2$. First, given a free 1-hyperladder $F_1 = (f_{\boldsymbol{j}^1}) : \boldsymbol{X} \to \boldsymbol{Y}$, one has to construct a free 2-hyperladder $F_2 = (f_{\boldsymbol{j}^2}) : \boldsymbol{X} \to \boldsymbol{Y}$ by means of F_1. Let $\boldsymbol{j}^2 = (j_1, j_2, j_3) \in \boldsymbol{J}(2)$. Put $t_1 = j_1$ and $t_2 = j_3$, and consider the 1-ladder $f_{\boldsymbol{t}^1} \in F_1$, where $\boldsymbol{t}^1 = (t_1, t_2) \in \boldsymbol{J}(1)$. Now define the 2-ladder $f_{\boldsymbol{j}^2} : \boldsymbol{X} \to \boldsymbol{Y}$ by using the maximal admissible restriction of $f_{\boldsymbol{t}^1}$ to first and second block of $\boldsymbol{j}^2$. It is readily seen that this construction preserves the identities and composition. If $n' > n > 1$, the construction is quite similar (but* not *unique). Let us observe that these functors do* not *preserve the homotopy relation. Namely, if they would preserve it, then all the S_n- and S^+_n-equivalences would coincide, contradicting to the known examples.*

5. The category characterization of the S-equivalence

Let us observe that there exists a sequential category $\underline{\mathcal{M}}$ having the "coordinates" all the categories $\underline{\mathcal{M}}_n$, $n \in \mathbb{N}$. That means, $Ob(\underline{\mathcal{M}}) = Ob(\underline{\mathcal{M}}_n)$, for any n, i.e. the objects are all inverse sequences of compacta, while

$$\underline{\mathcal{M}}(\boldsymbol{X}, \boldsymbol{Y}) = \{F = (F_n) \mid F_n \in \underline{\mathcal{M}}_n(\boldsymbol{X}, \boldsymbol{Y}) = \boldsymbol{L}_n(\boldsymbol{X}, \boldsymbol{Y}), n \in \mathbb{N}\}.$$

The composition is defined coordinatewise, and the identity on an $\boldsymbol{X}$ is $1_{\boldsymbol{X}} = (1_{\boldsymbol{X}n})$. There also exists its full subcategory $\underline{\mathcal{A}} \subseteq \underline{\mathcal{M}}$ determined by all the inverse sequences of compact ANR's (or compact polyhedra).

Let $F = (F_n), F' = (F'_n) \in \underline{\mathcal{M}}(\boldsymbol{X}, \boldsymbol{Y})$. Then F is said to be **homotopic to** F', denoted by $F \simeq F'$, provided $F_n \simeq F'_n$ in $\boldsymbol{L}_n(\boldsymbol{X}, \boldsymbol{Y})$, for every $n \in \mathbb{N}$. Clearly, this homotopy relation is an equivalence relation on every set $\underline{\mathcal{M}}(\boldsymbol{X}, \boldsymbol{Y})$. The equivalence class $[F]$ of an F is denoted by $\boldsymbol{F}$.

Theorem 7. *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be inverse sequences in $c\mathcal{M}$. Then $S(\boldsymbol{Y}) = S(\boldsymbol{X})$ if and only if there exists a pair of morphisms $F \in \underline{\mathcal{M}}(\boldsymbol{X}, \boldsymbol{Y})$, $G \in \underline{\mathcal{M}}(\boldsymbol{Y}, \boldsymbol{X})$ such that $GF \simeq 1_{\boldsymbol{X}}$ and $FG \simeq 1_{\boldsymbol{Y}}$ in $\underline{\mathcal{M}}$. Let X and Y be compact metric spaces, and let $\boldsymbol{X}$, $\boldsymbol{Y}$ be associated with X, Y respectively. Then $S(Y) = S(X)$ if and only if* *there exists a pair of morphisms* $F \in \underline{\mathcal{A}}(\boldsymbol{X},\boldsymbol{Y})$, $G \in \underline{\mathcal{A}}(\boldsymbol{Y},\boldsymbol{X})$ *such that* $GF \simeq 1_{\boldsymbol{X}}$ *and* $FG \simeq 1_{\boldsymbol{Y}}$ *in* $\underline{\mathcal{A}}$.

Proof. Recall that $S(\boldsymbol{Y}) = S(\boldsymbol{X})$ is equivalent to $S_n(\boldsymbol{Y}) = S(\boldsymbol{X})_n$ (or $S_n^+(\boldsymbol{Y}) = S^+(\boldsymbol{X})_n$) for every $n \in \{0\} \cup \mathbb{N}$. Notice that one may formally reduce the statement to $n \geq n_0$, for an arbitrary n_0. According to Definition 4., Lemmata 3. and 4. imply that, for every $n \in \mathbb{N}$, there exists a pair of morphisms $F_n \in \underline{\mathcal{M}}_n(\boldsymbol{X},\boldsymbol{Y})$, $G_n \in \underline{\mathcal{M}}_n(\boldsymbol{Y},\boldsymbol{X})$ such that $G_nF_n \simeq 1_{\boldsymbol{X}n}$ and $F_nG_n \simeq 1_{\boldsymbol{Y}n}$. Therefore, there exist an $F = (F_n) \in \underline{\mathcal{M}}(\boldsymbol{X},\boldsymbol{Y})$ and a $G = (G_n) \in \underline{\mathcal{M}}(\boldsymbol{Y},\boldsymbol{X})$ such that $GF \simeq 1_{\boldsymbol{X}}$ and $FG \simeq 1_{\boldsymbol{Y}}$ in $\underline{\mathcal{M}}$. Conversely, if $GF \simeq 1_{\boldsymbol{X}}$ and $FG \simeq 1_{\boldsymbol{Y}}$ in $\underline{\mathcal{M}}$, then $G_nF_n \simeq 1_{\boldsymbol{X}n}$ and $F_nG_n \simeq 1_{\boldsymbol{Y}n}$ in $\underline{\mathcal{M}}_n$, for every $n \in \mathbb{N}$. Now, according to Definition 4., Lemmata 6. and 7. imply $S_n(\boldsymbol{Y}) = S_n(\boldsymbol{X})$, for every $n \in \mathbb{N}$. Thus, $S(\boldsymbol{Y}) = S(\boldsymbol{X})$. The claim concerning compacta follows by Theorems 3. and 4. as well as by Theorems 5. and 6. □

An elegant reformulation of Theorem 7. is as follows:

Corollary 2. *Let* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *be inverse sequences in* $c\mathcal{M}$. *Then* $S(\boldsymbol{Y}) = S(\boldsymbol{X})$ *if and only if* $\boldsymbol{X} \leftrightarrow \boldsymbol{Y}$, *i.e.* $\boldsymbol{X}$ *is alike* $\boldsymbol{Y}$. *Consequently, for a pair* X, Y *of compacta,* $S(Y) = S(X)$ *if and only if* $X \leftrightarrow Y$, *i.e.* X *is alike* Y.

Remark 4. *(a) The established category characterization of the* S*-equivalence is* not *full, i.e. it is not described by means of the isomorphisms. Thus, there is no desired functor of the shape category to* $\underline{\mathcal{A}}$.

(b) It is clear that, in general, the homotopy relation $F \simeq F'$ *in* $\underline{\mathcal{M}}$ $(\underline{\mathcal{A}})$ *is* not *compatible with the category composition. Nevertheless, since the* S*-equivalence is an equivalence relation, Theorem 7. (Corollary 2.) implies that the alikeness is an equivalence relation too. Thus, the following useful fact holds: If* $F \in \underline{\mathcal{M}}(\boldsymbol{X},\boldsymbol{Y})$, $G \in \underline{\mathcal{M}}(\boldsymbol{Y},\boldsymbol{X})$, $F' \in \underline{\mathcal{M}}(\boldsymbol{Y},\boldsymbol{Z})$ *and* $G' \in \underline{\mathcal{M}}(\boldsymbol{Z},\boldsymbol{Y})$ *such that* $GF \simeq 1_{\boldsymbol{X}}$, $FG \simeq 1_{\boldsymbol{Y}}$, $G'F' \simeq 1_{\boldsymbol{Y}}$ *and* $F'G' \simeq 1_{\boldsymbol{Z}}$ *in* $\underline{\mathcal{M}}$, *then there exist an* $F'' \in \underline{\mathcal{M}}(\boldsymbol{X},\boldsymbol{Z})$ *and a* $G'' \in \underline{\mathcal{M}}(\boldsymbol{Z},\boldsymbol{X})$ *such that* $G''F'' \simeq 1_{\boldsymbol{X}}$ *and* $F''G'' \simeq 1_{\boldsymbol{Z}}$ *in* $\underline{\mathcal{M}}$.

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